Rank of a matrix

In this section we will deal with both row vectors and column vectors.

If A is an $m \times n$ matrix, the rows are n-vectors and the columns are m-vectors.

Def: The column space, $\operatorname{col} A$, of A is the subspace of $\mathbb{R}^m$ spanned by the columns of A.

The row space, $\operatorname{row} A$, of A is the subspace of $\mathbb{R}^n$ spanned by the rows of A.

Ex: If $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$,

$$\operatorname{col} A = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} = \mathbb{R}^2$$

$\operatorname{row} A = \operatorname{span}\{(1,2,1), (0,1,-1)\}$, a 2-dimensional subspace of $\mathbb{R}^3$, i.e. a plane.

Thm: Let A and B be $m \times n$ matrices.

1.) If $A \to B$ by elementary row operations, then $\operatorname{row} A = \operatorname{row} B$.

2.) If $A \to B$ by elementary column operations, then $\operatorname{col} A = \operatorname{col} B$.

This means, for example, that we can first put a matrix in row-echelon form in order to more easily find a basis for row A:

Ex: $A = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 0 & 0 & -2 \\ 0 & 2 & 2 & 8 \end{bmatrix}$.

To find row A, we perform row operations:

$$A \rightarrow \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -2 & -2 & -8 \\ 0 & 2 & 2 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -2 & -2 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So a basis for rowA is $\{(1,1,1,3),(0,1,1,4)\}$.

We can't use the same strategy to find colA, since we performed row operations to get to the RE form. However, the RE form tells us which columns of A we need in order to form a basis.

(First, recall that the rank of A is the # of leading ones in RE form.)

Theorem: Let A be an $m \times n$ matrix of rank r. Then

$$\dim(\text{col}A) = \dim(\text{row}A) = r. \quad (\text{so } r \leq m,\ r \leq n)$$

And if R is a row-echelon matrix such that

$$A \to R$$

by elementary row operations, then

1.) The r nonzero rows of R are a basis of row A.

2.) If the leading 1s lie in columns $j_1, j_2, \ldots, j_r$ of R, then columns $j_1, \ldots, j_r$ of $\underline{\underline{A}}$ are a basis of col A.

Ex: If $A = \begin{bmatrix} 1 & 0 & 3 & -2 \\ 2 & 1 & 2 & -1 \\ 1 & 1 & -1 & 1 \end{bmatrix}$

we can row reduce and get

$$\begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -4 & 3 \\ 0 & 1 & -4 & 3 \end{bmatrix} \longrightarrow \left.\begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right\} \text{basis for row}(A)$$

So $\operatorname{rank} A = 2$

The nonzero rows $\{(1,0,3,-2), (0,1,-4,3)\}$ are a basis for row(A).

The leading ones are in columns one and two, so a basis for col (A) is cols 1 and 2 of $\underline{\underline{A}}$ (not of RR matrix). So

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is a basis for col(A).

Recall that the image of a matrix A is vectors of the form $A\vec{x}$. We showed that

$$\text{im}(A) = \text{span of columns of } A$$

Thus $\text{im}(A) = \text{col}(A)$. Thus, the above gives us a method for finding a basis for $\text{im}(A)$, and we know now that

$$\dim(\text{im}(A)) = \text{rank}(A).$$

Recall that $\text{null}(A)$ is the vectors $\vec{x}$ such that $A\vec{x} = \vec{0}$. We showed that $\text{null}(A)$ is spanned by the basic solutions of $A\vec{x} = \vec{0}$. Infact, these actually form a basis.

So if A is an $m \times n$ matrix of rank r, there are $n-r$ basis solutions of $A\vec{x} = \vec{0}$. Thus,

$$\dim(\text{null}(A)) = n - r.$$

Ex: If $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 0 & -2 & -1 \\ -1 & -2 & -3 \end{bmatrix}$

We want to find the dimensions of $\text{null}(A)$ and $\text{im}(A)$ and bases for each.

First we set up an augmented matrix for $A\vec{x} = \vec{0}$.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & -2 & -1 & 0 \\ -1 & -2 & -3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So $im(A) = col(A)$ has dim 2 and basis

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \\ -2 \end{bmatrix} \right\}.$$

so $\dim(null(A)) = 3 - 2 = 1.$

Solving the system: $x + 2z = 0 \Rightarrow x = -2z$

$y + \frac{1}{2} z = 0 \Rightarrow y = \frac{-1}{2} z$

setting $t = z$:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2t \\ -\frac{1}{2}t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ -\frac{1}{2} \\ 1 \end{bmatrix},$$

so $\left\{ \begin{bmatrix} -2 \\ -\frac{1}{2} \\ 1 \end{bmatrix} \right\}$ is a basis for $null(A)$.

We now have a better method for finding the dimension of a subspace:

Ex: (5.4.2b)

$$U = span \left\{ \begin{bmatrix} 1 \\ 5 \\ -6 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ -8 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ -10 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \\ 12 \end{bmatrix} \right\}$$

Set $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ -6 & -8 & -10 & 12 \end{bmatrix}$ Then $U = col(A)$

$$\rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & 4 & 8 & 36 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & -16 \\ 0 & 0 & 0 & 24 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

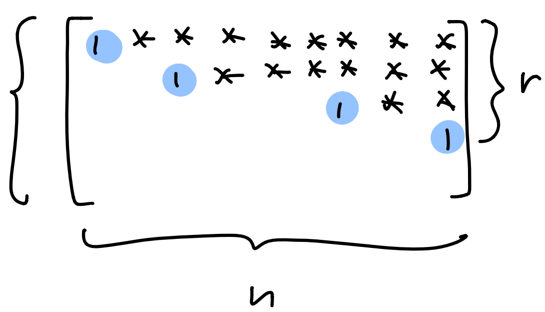
columns 1, 2, 4 of A form a basis

$\dim(U) = \operatorname{rank}(A) = 3$

basis for $U = \left\{ \begin{bmatrix} 1 \\ 5 \\ -6 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ -8 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \\ 12 \end{bmatrix} \right\}$

To summarize: If A is an $m \times n$ matrix, and R = RE form of A

rank(A) = # leading 1s.



col(A) = span of cols of A (subspace of $\mathbb{R}^m$)

- $\dim(\operatorname{col}(A)) = \operatorname{rank}(A)$
- To find a basis: columns of A, corresponding to leading ones in R.

im(A) = vectors in $\mathbb{R}^m$ of form $A\vec{x}$.

- $\operatorname{im}(A) = \operatorname{col}(A)$

$\operatorname{row}(A)$ = span of rows of A (subspace of $\mathbb{R}^n$)

- $\dim(\operatorname{row}(A)) = \operatorname{rank}(A)$
- basis = nonzero rows of R

$\operatorname{null}(A)$ = solutions to $A\vec{x} = \vec{0}$ (subspace of $\mathbb{R}^n$)

- $\dim(\operatorname{null}(A)) = n - \operatorname{rank}(A)$

  = # of nonleading variables in R.
- basis = basic solutions of $A\vec{x} = \vec{0}$

Practice problems: 5.4: 1, 2, 3, 7